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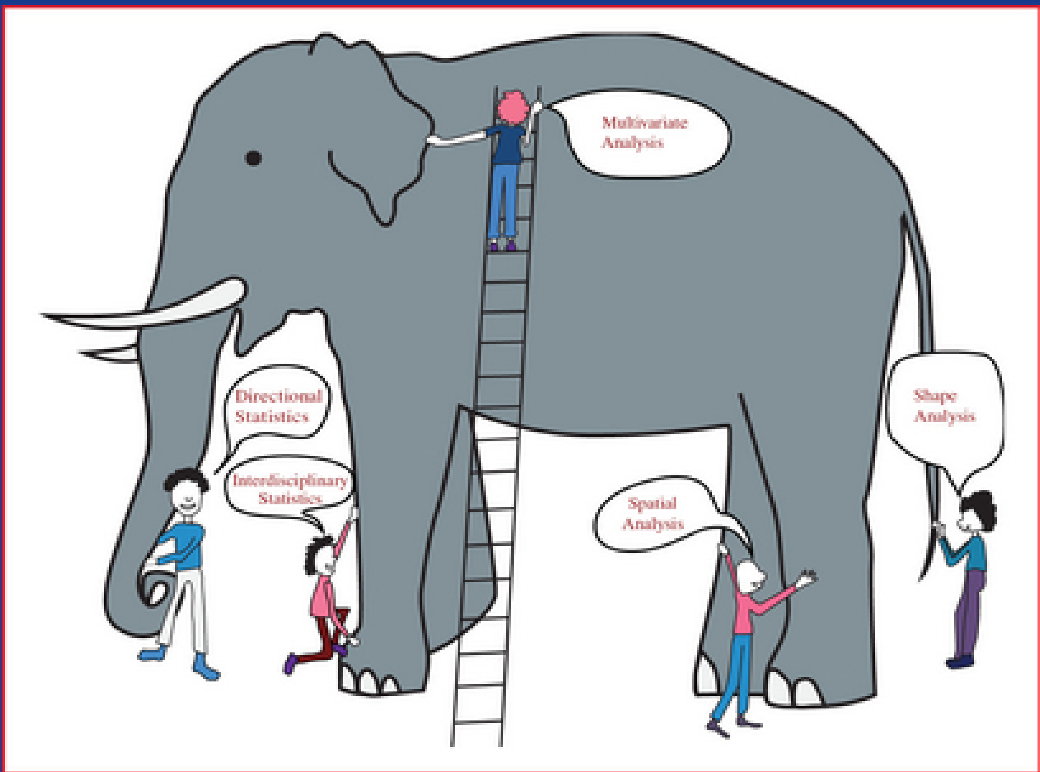

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6

On two-sample tests for circular data based on spacing-frequencies

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6.1 Introduction

In many scientific disciplines, observations are directions and are referred to as "directional data". A two-dimensional direction can be represented by (i) a vector in $\mathbb{R}^2$ of length one since magnitude has no relevance, (ii) by a complex number of unit modulus, (iii) by a point of S^1, the circumference of the unit circle centered at the origin, or (iv) by an angle measured in radians or degrees. In this chapter, we adopt this last representation using radians. Data representing two-dimensional directions is referred as "circular data." Circular data arise in many natural sciences, including geology, seismology, meteorology, animal behavior, and so on just to name a few. Moreover, any periodic phenomenon with a known period can be represented in terms of two-dimensional directions, such as the circadian rhythms.

The analysis of circular data relies on specific statistical procedures, which differ from usual statistical methodology for the real line. Since there is no prescribed null direction or sense of rotation (either clockwise or anticlockwise), it is important that procedures for circular data remain independent of the arbitrary choices of the zero direction and of the sense of rotation. The von Mises distribution provides one of the basic models for circular

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data. It is often considered as central as the normal distribution is, for linear data. However, since there is no systematic mathematical rationale for invoking the von Mises distribution as much as there is for using a normal distribution on the line, distribution-free or nonparametric techniques assume a more important role in the context of circular data. This chapter focuses on nonparametric tests for circular data and in particular on nonparametric two-sample tests based on the so-called "spacing-frequencies". In this chapter, the importance of this type of tests is stressed in terms of invariance properties. Moreover, tests based on "circular ranks" on the circle can be reexpressed in terms of these spacing-frequencies.

Two seminal publications on circular distributions are Langevin (1905) and Lévy (1939) and one pioneering statistical analysis of directional data is due to Fisher (1953). Two general references are Mardia and Jupp (2000) and Jammalamadaka and SenGupta (2001). There is considerable literature on modeling and analysis of circular data including, for example, Rao (1969) and Gatto and Jammalamadaka (2007).

The remaining part of this chapter is organized as follows. Section 6.2 presents an overview of spacing-frequencies tests for circular data. In particular, it presents some careful analysis of the invariance, the maximality, and the symmetry properties. It then reviews three well-known two-sample tests for circular data, which are the Dixon, the Wheeler–Watson, and the Wald–Wolfowitz tests. A slight generalization based on high-order spacing-frequencies, called multispacing-frequencies, is then reviewed. The end of Section 6.2 mentions a conditional representation for the distribution of the multispacing-frequencies, which allows one to derive the asymptotic normality and a saddlepoint approximation. Section 6.3 provides an extension of Rao's one-sample spacings test (see Rao 1969, 1976) to the two-sample setting using the spacing-frequencies. A geometrical interpretation of the proposed test statistic is provided. Its exact distribution and a saddlepoint approximation are then discussed. Section 6.4 provides a Monte Carlo comparison of the powers of Wheeler–Watson's, Dixon's and Rao's two-sample spacing-frequencies tests. In this study, it is demonstrated that if one of the two samples is suspected of coming from a certain bimodal distribution, Rao's and Dixons's spacing-frequencies tests have comparable power, whereas Wheeler–Watson test, which is commonly used in this context, has substantially lower power. It may be remarked that this deficiency is comparable to that suffered by Rayleigh's test for uniformity in a single sample, when the data is suspected of not being unimodal.

6.2 Spacing-frequencies tests for circular data

Suppose we have two independent samples of circular data, the first sample consisting of m independent and identically distributed (iid) circular random variables $X_1, \ldots, X_m$, with probability distribution P_X and a second sample of n iid circular random variables $Y_1, \ldots, Y_n$, with probability distribution P_Y. As mentioned, these samples represent angles in radians, with respect to some arbitrary origin and sense of rotation. P_X and P_Y are circular distributions in the sense that they assign total measure one to $[c, c + 2\pi)$, $\forall c \in \mathbb{R}$. The general two-sample problem is to test the null hypothesis that both these samples come from the same parent population, viz.

$$\mathrm{H}_0 \colon P_X = P_Y. \tag{6.1}$$

Stating H_0 in terms of the probability distributions or measures P_X and P_Y, instead of the usual formulation in terms of cumulative distribution functions (cdf), is more appropriate because the cdf depends on the choice of the null direction and the sense of rotation. For convenience, we denote $X = (X_1, \ldots, X_m)$ and $Y = (Y_1, \ldots, Y_n)$.

6.2.1 Invariance, maximality and symmetries

Let $X_{(1)} \leq \cdots \leq X_{(m)}$ denote the circularly ordered values $X_1, \ldots, X_m$, for a given origin and sense of rotation. With $\mathrm{I}\{A\}$ denoting the indicator of statement A, the random counts

$$S_j = \sum_{i=1}^{n} \mathrm{I}\{Y_i \in [X_{(j)}, X_{(j+1)})\}, \text{ for } j = 1, \ldots, m-1, \text{ and } S_m = n - \sum_{j=1}^{m-1} S_j,$$

are commonly called (circular) spacing-frequencies, as they provide the number of observations $Y_1, \ldots, Y_n$ which lie in-between successive gaps made by $X_{(1)}, \ldots, X_{(m)}$. A substantial amount of nonparametric theory for the real line is based on the "ranks," for example, refer to Sidak et al. (1999). If one were to define "ranks" on the circle with respect to the same origin and sense of rotation (on which they depend), then the spacing-frequencies $S_1, \ldots, S_m$ could be related to such ranks. Specifically, if R_k denotes the circular rank of the k^{th} largest $X_1, \ldots, X_m$ in the combined sample, with origin given by $X_{(1)}$ and same sense of rotation as before, then

$$R_k = k + \sum_{j=1}^{k-1} S_j, \quad \text{for } k = 1, \ldots, m, \tag{6.2}$$

(where $\sum_{j=1}^{0} \stackrel{\text{def}}{=} 0$). Conversely,

$$R_{k+1} = R_k + S_k + 1, \text{ for } k = 1, \ldots, m-1, \text{ and } R_m = m + n - S_m$$

yield

$$S_k = R_{k+1} - R_k - 1, \text{ for } k = 1, \ldots, m-1, \text{ and } S_m = m + n - R_m, \tag{6.3}$$

so that, $S_1, \ldots, S_m$ may be thought of "rank-differences" when such ranks are well defined, as they are on the line. Moreover, note that in this context, the spacing-frequencies are well defined even in the presence of ties, that is, repeated values in the combined sample. Indeed, there is no reason to assume absolute continuity (with respect to the Lebesgue measure) of either P_X or P_Y, whereas ranks have to be adapted whenever ties have positive probability of occurring, for example, by defining "midranks."

A natural question that arises in this context is the symmetry with respect to roles of the two samples X and Y in the construction of the spacing-frequencies tests. Precisely, let $Y_{(1)} \leq \cdots \leq Y_{(n)}$ denote the circularly ordered values $Y_1, \ldots, Y_n$, for the same origin and sense of rotation used with $S_1, \ldots, S_m$. The random counts

$$S'_j = \sum_{i=1}^{m} \mathrm{I}\{X_i \in [Y_{(j)}, Y_{(j+1)})\}, \text{ for } j = 1, \ldots, n-1, \text{ and } S'_n = m - \sum_{j=1}^{n-1} S'_j,$$

are called the "dual spacing-frequencies." The next proposition addresses this question of sample symmetry.

Proposition 1
The dual spacing-frequencies $S'_1, \ldots, S'_n$ can be obtained as a one-to-one function of the original spacing-frequencies $S_1, \ldots, S_m$ and conversely, so that tests may be based on either set of spacing-frequencies.

We show this result in case where P_X and P_Y are absolutely continuous.

Proof
Assume P_X and P_Y absolutely continuous. Let R'_k denote the circular rank of the k^{th} largest $Y_1, \ldots, Y_n$ in the combined sample, for $k = 1, \ldots, n$, with origin given by $X_{(1)}$, which is the origin used for the original ranks, and same rotation sense as for the original ranks. Then, we can compute the dual circular ranks as follows:

$$R'_1 = 1 + \sum_{k=1}^{m} \mathrm{I}\{X_k \in [X_{(1)}, Y_{(1)})\} \text{ and } R'_k = k + R'_1 - 1 + \sum_{j=1}^{k-1} S'_j, \text{ for } k = 2, \ldots, n. \tag{6.4}$$

Given absolute continuity, we have

$$\{R'_1, \ldots, R'_n\} = \{1, \ldots, m+n\} \backslash \{R_1, \ldots, R_m\},$$

where the elements of the aforementioned sets are ordered from the smallest to the largest, when going from left to right. We then obtain

$$S'_k = R'_{k+1} - R'_k - 1, \text{ for } k = 1, \ldots, n-1, \text{ and } S'_n = m + n - R'_n + R'_1.$$

Conversely, absolute continuity yields

$$\{R_1, \ldots, R_m\} = \{1, \ldots, m+n\} \backslash \{R'_1, \ldots, R'_n\}$$

and $S_1, \ldots, S_m$ can be obtained through (6.3). □

We can thus arbitrarily decide which sample is used for constructing the spacings and which sample is used for obtaining the frequencies. Constructing tests based on either set of spacing-frequencies would make sense.

It turns out that the spacing-frequencies play a central role in comparing two circular distributions. This is because in many applied problems with circular data, the null direction and the sense of rotation are arbitrarily chosen. Assume that all circular random variables take values on $[0, 2\pi)$ and denote by $\mathcal{G}$ the transformation group consisting of all changes of origin (zero direction) and of the two changes of sense of rotation $[0, 2\pi)^{m+n} \to [0, 2\pi)^{m+n}$, that is, for the two samples. We recall that a (two-sample test) statistic $T : [0, 2\pi)^{m+n} \to \mathbb{R}$ is called invariant with respect to the transformation group $\mathcal{G}$ if, for any (X, Y) and $(\tilde{X}, \tilde{Y})$ $[0, 2\pi)^{(m+n)}$,

$$\exists\, g \in \mathcal{G} \text{ such that } (\tilde{X}, \tilde{Y}) = g(X, Y) \Longrightarrow T(\tilde{X}, \tilde{Y}) = T(X, Y).$$

If, in addition to this, for any (X, Y) and $(\tilde{X}, \tilde{Y})$ $[0, 2\pi)^{(m+n)}$,

$$T(\tilde{X}, \tilde{Y}) = T(X, Y) \Longrightarrow \exists\, g \in \mathcal{G} \text{ such that } (\tilde{X}, \tilde{Y}) = g(X, Y),$$

then the statistic T is a "maximal invariant". It can then be shown that the statistic T is $\mathcal{G}$-invariant iff T is a function of maximal $\mathcal{G}$-invariant. This leads us to ask whether

$(S_1 \ldots, S_m)$ is invariant or maximal invariant with respect to the transformation group $\mathcal{G}$ and for the testing problem (6.1).

Consider first the equivalence classes generated by any maximal invariant for $\mathcal{G}$, cf. Schach (1969).

Proposition 2
The circular $[0, 2\pi)^{m+n}$-valued samples (X, Y) and $(\tilde{X}, \tilde{Y})$ belong to the same equivalence class generated by $\mathcal{G}$, iff

$$(S_1, \ldots, S_m) = (\tilde{S}_{1+k}, \ldots, \tilde{S}_{m+k}), \text{ for some } k \in \{0, \ldots, m-1\},$$

with $\tilde{S}_j = \tilde{S}_{j-m}$, whenever $j > m$, or

$$(S_1, \ldots, S_m) = (\tilde{S}_m, \ldots, \tilde{S}_1),$$

where $(S_1, \ldots, S_m)$ are the spacing-frequencies of (X, Y) and $(\tilde{S}_1, \ldots, \tilde{S}_m)$ are the spacing-frequencies of $(\tilde{X}, \tilde{Y})$.

We often use the terminology that (X, Y) and $(\tilde{X}, \tilde{Y})$ are equal modulo $\mathcal{G}$.

Proof
The transformation group of all changes of origin is made of the set of functions $\mathcal{F}_1$ $[0, 2\pi)^{m+n} \to [0, 2\pi)^{m+n}$, which transform the spacing-frequencies of (X, Y) as

$$(S_1, \ldots, S_m) \mapsto (S_2, \ldots, S_m, S_1).$$

The transformation group of sense reversions is made of the set of functions $\mathcal{F}_2$ $[0, 2\pi)^{m+n} \to [0, 2\pi)^{m+n}$ yielding

$$(S_1, \ldots, S_m) \mapsto (S_m, \ldots, S_1),$$

when clockwise changes to anticlockwise. The transformation group $\mathcal{G}$ is made of $\mathcal{F}_1 \cup \mathcal{F}_2$. So we clearly obtain the equivalence classes mentioned in Proposition 2. □

Theoretically, one can obtain the desired $\mathcal{G}$-invariance by taking, for example, the supremum or the average of any function of $S_1, \ldots, S_m$ over the given equivalence classes, but this approach seems clumsy and should not lead to any practical or useful statistic. Therefore, as a viable alternative, we consider functions of "ordered" $S_1, \ldots, S_m$, which serve almost the same purpose and lead to $\mathcal{G}$-invariance. Obviously, the vector $(S_1, \ldots, S_m)$ is not by itself $\mathcal{G}$-invariant: if we change for example the zero direction, then the new vector of spacing-frequencies is a permutation of the original one. So let $S_{(1)} \leq \cdots \leq S_{(m)}$ denote the ordered spacing-frequencies $S_1 \ldots, S_m$. They constitute an invariant statistic for $\mathcal{G}$ and so is any statistic based on these ordered values. The complete description is given by the next proposition.

Proposition 3
1. T is a symmetric function of $S_1, \ldots, S_m \Longleftrightarrow T$ is a function of $(S_{(1)}, \ldots, S_{(m)})$.
2. T is a symmetric function of $S_1, \ldots, S_m \Longrightarrow T$ is $\mathcal{G}$-invariant.

Proof
1. (⇒) T is a function of any permutation of $S_1, \ldots, S_m$ and in particular of $(S_{(1)}, \ldots, S_{(n)})$.
(⇐) T is invariant under permutations of $S_1, \ldots, S_m$, that is, T is a symmetric function of these values.
2. By part 1, T is a function of $(S_{(1)}, \ldots, S_{(m)})$. With Proposition 2, it is directly seen that any $\mathcal{G}$-transformation is without effect on these ordered values. □

We should remark that maximal invariance is, however, not obtained by $(S_{(1)}, \ldots, S_{(m)})$.

Proposition 4
The vector of ordered spacing-frequencies $(S_{(1)}, \ldots, S_{(m)})$ is not a maximal invariant statistic under the group $\mathcal{G}$.

Proof
Denote by $\tilde{S}_1, \ldots, \tilde{S}_m$ the spacing-frequencies obtained by the new samples $\tilde{X}$ and $\tilde{Y}$. Denote also $\tilde{S}_{(1)} \leq \cdots \leq \tilde{S}_{(m)}$ the corresponding ordered spacing-frequencies. "Maximality" means that

$$\tilde{S}_{(k)} = S_{(k)}, \text{ for } k = 1, \ldots, m \implies (\tilde{X}, \tilde{Y}) = g(X, Y), \quad \text{for some } g \in \mathcal{G}.$$

However, $S_{(k)} = \tilde{S}_{(k)}$, for $k = 1, \ldots, m$, means exactly that $(\tilde{S}_1, \ldots, \tilde{S}_m)$ is obtained through a permutation of the elements of $(S_1, \ldots, S_m)$. This last situation can be obtained in many different ways: for example, with $\tilde{X} = X$ and with $\tilde{Y}$ obtained from different individual transforms of the elements of Y, in such a way that $(\tilde{S}_1, \ldots, \tilde{S}_m)$ becomes the desired permutation. It is then not necessary that $\tilde{X}$ and $\tilde{Y}$ derive from a change of origin or sense of rotation, applied to X and Y simultaneously. Thus, we do not have maximality. □

As a concrete counter-example, the $\mathcal{G}$-invariant Wheeler–Watson statistic can be reexpressed as a function of $(S_1, \ldots, S_m)$, but not as a function of $(S_{(1)}, \ldots, S_{(m)})$: it is not a symmetric function of the spacing-frequencies. See Example 6 for details.

We may note the following observations about the unordered spacing-frequencies. First, $S_1, \ldots, S_m$ are exchangeable random variables under H_0 (i.e., any permutation of these random variables is equiprobable and follows the Bose–Einstein distribution in statistical mechanics). Second, consider any class of circular models parameterized by the null direction and by the sense of rotation. Then, $(S_1, \ldots, S_m)$ is an ancillary statistic for this class of models under H_0, that is, its distribution is invariant within this class.

6.2.2 An invariant class of spacing-frequencies tests

From the previous results, because the popular nonparametric Wilcoxon test statistic takes the nonsymmetric form $\sum_{k=1}^{m} kS_k$, it should not be used with circular data. Define $\mathbb{N} = \{0, 1, \ldots\}$. Assume $h : \mathbb{N} \to \mathbb{R}$ and $h_j : \mathbb{N} \to \mathbb{R}$, for $j = 1, \ldots, m$, satisfy certain mild regularity conditions. Holst and Rao (1980) consider nonparametric test statistics of the form

$$T_{m,n} = \sum_{j=1}^{m} h(S_j) \quad \text{and} \quad T^*_{m,n} = \sum_{j=1}^{m} h_j(S_j), \tag{6.5}$$

which are called the symmetric and the nonsymmetric test statistics based on spacing-frequencies. As mentioned in Proposition 2, only the symmetric statistic $T_{m,n}$ is relevant with circular data, when considering $\mathcal{G}$-invariance. However, the asymptotic efficiencies of the nonsymmetric tests $T^*_{m,n}$ are shown to be superior by Holst and Rao (1980), when considering data on the real line.

The limiting null distribution of the most general nonsymmetric statistic $T^*_{m_\nu,n_\nu}$, when $\{m_\nu\}_{\nu\geq 0}$ and $\{n_\nu\}_{\nu\geq 0}$ are nondecreasing sequences in $\mathbb{N}^\infty$ such that, as $\nu \to \infty$,

$$m_\nu \to \infty,\ n_\nu \to \infty \ \text{ and } \ \rho_\nu \stackrel{\text{def}}{=} \frac{m_\nu}{n_\nu} \to \rho, \quad \text{for some } \rho \in (0,\infty), \tag{6.6}$$

is given by

$$\frac{\sum_{j=1}^{m_\nu} h_j(S_j) - \mu_{m_\nu}}{\sigma_{m_\nu}} \xrightarrow{\text{d}} \mathcal{N}(0,1),$$

where μ_m and σ^2_m are defined as follows. If $V_1,\ldots,V_m$ are i.i.d. geometric random variables with

$$\mathsf{P}[V_1 = k] = \left(\frac{1}{1+\rho}\right)^k \frac{\rho}{1+\rho}, \text{ for } k = 0,1,\ldots, \tag{6.7}$$

then $\mu_m = \mathsf{E}[\sum_{j=1}^m h_j(V_j)]$ and $\sigma^2_m = \text{var}(\sum_{j=1}^m h_j(V_j) - \beta_m \sum_{i=j}^m V_j)$, in which $\beta_m = \text{cov}(\sum_{j=1}^m h_j(V_j), \sum_{j=1}^m V_j)/\text{var}(\sum_{j=1}^m V_j)$; refer to Corollary 3.1 on p. 41 of Holst and Rao (1980).

One can see that the circular Wald and Wolfowitz (1940) run test (see Example 6) and the circular Dixon (1940) test (see Example 5) have the symmetric form $T_{m,n}$, whereas the Wheeler and Watson (1964) test (see Example 7) is nonsymmetric with respect to the spacing-frequencies. One can also note that any linear function of the ranks $R_1,\ldots,R_m$ in the combined sample can be expressed in terms of the nonsymmetric statistic $T_{m,n}$. Further discussion on this type of tests can be found in Rao and Mardia (1980).

We now give two examples of symmetric statistics of the form given in (6.5). A third example will be suggested later in Section 6.3.1. Then we present Wheeler–Watson test, which will be analyzed numerically in Section 6.4.

Example 5 *Dixon's test* Theorem 4.2 at p. 48 of Holst and Rao (1980) states that the locally most powerful test among all symmetric tests in the spacing-frequencies given in (6.5) is

$$T_{m,n} = \sum_{j=1}^m S_j^2. \tag{6.8}$$

Note that this local optimality is under a sequence of alternative cdf's for Y_1 that converge to the cdf of X_1, both depending on the choices of zero direction and the sense of rotation, see Equation (4.2) in Holst and Rao (1980).

Example 6 *Wald–Wolfowitz run test* Another example in the class of symmetric two-sample test statistics is given by the circular version of Wald–Wolfowitz run test statistic; see also David and Barton (1962). The Wald–Wolfowitz run test statistic is $T_{m,n}$

as given by (6.5) with $h(x) = \mathrm{I}\{x > 0\}$. We define a "$Y$-run" in the combined sample as the largest nonempty group of adjacent Y-values. Since any positive value of $S_1, \ldots, S_m$ constitute a Y-run, $T_{m,n}$ gives the number of Y-runs in the combined sample, and it takes values in $\{1, \ldots, m\}$. But in the circle, the number of X- and Y-runs must be same and so $2T_{m,n}$ gives the total number of runs made by the combined sample. Large values of $T_{m,n}$ show evidence for equal spread, that is, for H_0. Note that Section 2.3 of Gatto (2000) provides a saddlepoint approximation to the distribution of this statistic under H_0, in the linear setting.

Example 7 *Wheeler–Watson test* This test has also been called the Mardia–Watson–Wheeler test, see e.g. p. 101 of Batschelet (1981), and the uniform scores test. It assumes absolute continuity of P_X and P_Y (in order to almost surely exclude ties). The idea is the following. Adjust the values of X and Y by respecting their relative order, in such a way to obtain $m+n$ equidistant values. So the spacings between any two consecutive adjusted values are all equal and equal to $2\pi/(m+n)$. For a given choice of origin and rotation sense, X and Y are thus mapped onto $\{2\pi k/n\}_{k=1,\ldots,m+n}$. The values of X become $2\pi R_1/(m+n), \ldots, 2\pi R_m/(m+n)$, which are called "uniform scores," where $R_1, \ldots, R_m$ are, as before, the ranks of X in the combined sample. Because of being uniformly spread, the overall resultant vector V of the uniform scores is null, that is, $V = 0$. However, since $V = V_X + V_Y$, where V_X and V_Y are the resultant vectors of the transformed samples X and Y, it follows that $V_X = -V_Y$. (So only one of the statistics V_X and V_Y is relevant.) Under H_0, the two samples should be evenly spread over the circumference and thus $||V_X|| \simeq ||V_Y|| \simeq 0$. So a relevant decision rule is given by: reject H_0 if $||V_X||$ is large. But V_X can be obtained from the spacing-frequencies through (6.4),

$$||V_X||^2 = \left\{\sum_{k=1}^{m} \cos\left(\frac{2\pi}{m+n} R_k\right)\right\}^2 + \left\{\sum_{k=1}^{m} \sin\left(\frac{2\pi}{m+n} R_k\right)\right\}^2$$

$$= \left\{\sum_{k=1}^{m} \cos\left(\frac{2\pi}{m+n}\left[k + \sum_{j=1}^{k-1} S_j\right]\right)\right\}^2 + \left\{\sum_{k=1}^{m} \sin\left(\frac{2\pi}{m+n}\left[k + \sum_{j=1}^{k-1} S_j\right]\right)\right\}^2, \tag{6.9}$$

which cannot have the symmetric form $T_{m,n}$ given in (6.5). From Proposition 3, it is not a function of $(S_{(1)}, \ldots, S_{(m)})$. However, $||V_X||$ is clearly $\mathcal{G}$-invariant. This illustrates the non-maximality of $(S_{(1)}, \ldots, S_{(m)})$ claimed by Proposition 4.

Note, however, the following drawback inherent to this test in the presence of bimodal distributions. Assume that the sample Y presents two similar modes, the second mode being located approximately at the antimode. For various configurations of the sample X, these modes lead to the cancelation in the uniform scores so that $||V_Y||$ and $||V_X||$ tend to be small, even without H_0 being true. Low power is thus expected in these cases. Our extensive simulations in Section 6.4 provide a numerical confirmation. This weakness, as we remarked in the introduction, is similar to that suffered by Rayleigh's test for uniformity when used in bimodal or multimodal samples.

6.2.3 Multispacing-frequencies tests

It turns out that the asymptotic power of the tests based on spacing-frequencies (6.5) can be improved by considering larger spacings or gaps in the following sense. Let $l \geq 1$ denote the order of the gap between the values of X and define the nonoverlapping or disjoint multispacing-frequencies as

$$S_j^{(l)} = \sum_{i=1}^{n} \mathrm{I}\{Y_i \in [X_{(jl)}, X_{((j+1)l)})\}, \text{ for } j = 1, \dots, r, \quad \text{with } r \stackrel{\text{def}}{=} \left\lfloor \frac{m}{l} - 1 \right\rfloor.$$

So if $l = 1$, then $r = m - 1$ and $S_j^{(1)} = S_j$, for $j = 1, \dots, m - 1$. In this case, S_m can be defined as before.

Assume $h : \mathbb{N} \to \mathbb{R}$ and $h_j : \mathbb{N} \to \mathbb{R}$, for $j = 1, \dots, r$, satisfy certain regularity conditions (given under Assumption A in Jammalamadaka and Schweitzer 1985) and define the general classes of test statistics

$$T_{m,n}^{(l)} = \sum_{j=1}^{r} h(S_j^{(l)}) \quad \text{and} \quad T_{m,n}^{(l)*} = \sum_{j=1}^{r} h_j(S_j^{(l)}), \tag{6.10}$$

which represent, respectively, the symmetric and the nonsymmetric test statistics based on multispacing-frequencies. When $l = 1$, both sums in (6.10) go up to $m = r + 1$ (instead of r). Jammalamadaka and Schweitzer (1985) establish the asymptotic normality of these statistics (and of similar statistics based on overlapping multispacing-frequencies), under the null hypothesis and under asymptotically close alternatives as well. The locally most powerful test, for a given smooth sequence of alternative c.d.f. of Y_1 converging toward the cdf of X_1, is provided by Theorem 3.2 at pp. 41–42 of Jammalamadaka and Schweitzer (1985). We reject H_0 if

$$\sum_{j=1}^{r} g\left(\frac{j}{r+1}\right) S_j^{(l)} > c,$$

for some $c \in \mathbb{R}$, where the real-valued function g depends on the sequence of alternative cdf of Y_1 and on the cdf of X_1. So the optimal test statistic has the nonsymmetric form $T_{m,n}^{*}$ given in (6.10). For the same reason that nonsymmetric statistics in spacing-frequencies are not $\mathcal{G}$-invariant and symmetric statistics are $\mathcal{G}$-invariant, the nonsymmetric statistic in the multispacing-frequencies $T_{m,n}^{(l)*}$ is not $\mathcal{G}$-invariant, whereas the symmetric statistic $T_{m,n}^{(l)}$ is $\mathcal{G}$-invariant. Jammalamadaka and Schweitzer (1985) show that the sum of squared multispacing-frequencies, leading to the statistic

$$T_{m,n}^{(l)} = \sum_{j=1}^{r} \left(S_j^{(r)}\right)^2, \tag{6.11}$$

is the optimal choice among all symmetric and nonoverlapping statistics. When $l = 1$, this is the Dixon (1940) statistic of Example 5. We may note that the multispacing-frequencies statistics (6.10) are clearly nonsymmetric with respect to the roles given to the samples X and Y: if the spacings would be defined by Y and the frequencies by X, then we would obtain a different test statistic.

6.2.4 Conditional representation and computation of the null distribution

For the most general statistics based on the multispacing-frequencies, consider the independent random variables $W_1, \dots, W_r$ with the negative binomial distribution with parameters

l and $p = \rho/(1+\rho)$, namely

$$\mathsf{P}[W_1 = k] = \binom{l+k-1}{k} \left(\frac{\rho}{1+\rho}\right)^l \left(\frac{1}{1+\rho}\right)^k, \quad \text{for } k = 0, 1, \ldots \tag{6.12}$$

The next proposition tells that under H_0, the r multispacing-frequencies have the same distribution as these negative binomial random variables, when conditioned to sum up to n.

Proposition 8
If $W_1, \ldots, W_r$ are independent random variables with probability function (6.12), then $\forall \rho \in (0, \infty)$,

$$(S_1^{(l)}, \ldots, S_r^{(l)}) \sim (W_1, \ldots, W_r) \mid Z_r = n, \tag{6.13}$$

where $Z_r = \sum_{j=1}^{r} W_j$.

This conditional representation is the central argument for the determination of the null asymptotic distribution of symmetric statistics, based on (nonoverlapping) multispacing-frequencies. The next proposition is a direct consequence of Theorem 4.2 on pp. 613–614 of Jammalamadaka and Schweitzer (1985).

Proposition 9
The following asymptotic distribution holds under H_0 and under the asymptotics (6.6),

$$r^{-\frac{1}{2}} \sum_{j=1}^{r} \{h(S_j^{(l)}) - \mathsf{E}[h(W_1)]\} \xrightarrow{\mathrm{d}} \mathcal{N}(0, \zeta_l^2), \tag{6.14}$$

where

$$\zeta_l^2 = \mathrm{var}(h(W_1)) - \frac{\rho^2}{1+\rho} \cdot \mathrm{cov}^2(h(W_1), W_1). \tag{6.15}$$

We also note that the distributions of the most general test statistic $T_{m,n}^{(l)*}$ can be obtained with saddlepoint approximation suggested by Gatto and Jammalamadaka (2006), which also exploits the conditional representation (6.13); see also Section 6.3.3.

6.3 Rao's spacing-frequencies test for circular data

In this section, we provide an extension of the idea of Rao's one-sample spacings test (cf. Rao 1976) to the two-sample setting, making use of the spacing-frequencies. Although the Wheeler–Watson test is a popular two-sample nonparametric test, Rao's spacing-frequencies test has a simple intuitive interpretation and has efficiencies comparable to that of the locally most powerful Dixon's test. It also admits a nice geometrical interpretation, which is provided in Section 6.3.1. However, as mentioned in Example 6, Wheeler–Watson test has the drawback of not distinguishing the case where the P_X is bimodal at its antimode from H_0, a situation that often occurs when measuring wind directions, see for example, Section 3 of Gatto and Jammalamadaka (2007). The Wheeler–Watson test may have low power in this circumstance. A small sample power comparison in this situation and with these three tests, namely, the Wheeler–Watson test, the Dixon test, and the Rao spacing-frequencies test, is presented in Section 6.4.

6.3.1 Rao's test statistic and a geometric interpretation

Motivated by Rao's one-sample spacings test which takes the form $\sum_{j=1}^{m}|D_j - 1/m|$, where $D_1, \ldots, D_m$ denote the (one-sample) spacings (i.e., the gaps between successive points or the first-order differences) and which is widely used for testing isotropy of a single sample, we will define what we will call "Rao's two-sample spacing-frequencies test," by

$$T_{m,n} = \frac{1}{2}\sum_{j=1}^{m}\left|S_j - \frac{n}{m}\right|. \tag{6.16}$$

This is symmetric in the spacing-frequencies and has been briefly mentioned in the study by Rao and Mardia (1980).

An interesting geometrical interpretation can be given for this statistic similar to that available for the Rao's spacings test. We first note that

$$\sum_{j=1}^{m}\left(S_j - \frac{n}{m}\right) = 0 \implies T_{m,n} = \sum_{j=1}^{m}\max\left\{S_j - \frac{n}{m}, 0\right\}. \tag{6.17}$$

Consider for the moment a circle with circumference n (i.e.$n\mathrm{S}^1/(2\pi)$) and consider the spacing-frequencies $S_1, \ldots, S_m$ as spacings of a conceptual sample $Z = (Z_1, \ldots, Z_m)$ on this circle, that is, $S_j = Z_{(j+1)} - Z_{(j)}$, for $j = 1, \ldots, m-1$, and $S_m = Z_{(1)} - Z_{(m)}$. With this interpretation, we can consider the spacing-frequencies as $\{0, \ldots, n\}$-valued random variables. On this circle, we then place m arcs of equal length n/m, starting at each one of the m values of Z. In this situation, $T_{m,n}$ as given by (6.16) becomes the total "uncovered part of the circumference" of this circle. The case $T_{m,n} = 0$ means that all spacing-frequencies are exactly equal, that is,

$$S_1 = \cdots = S_m = \frac{n}{m},$$

which is clearly the strong evidence for $\mathrm{H}_0 : P_X = P_Y$. On the other extreme, the case $T_{m,n} = n(1 - 1/m)$ means that

$$\exists j \in \{1, \ldots, n\}, \text{ such that } S_j = n \text{ and } S_k = 0, \ \forall k \neq j \in \{1, \ldots, n\},$$

which is the strong evidence against H_0, that is, for dissimilarity between P_X and P_Y.

6.3.2 Exact distribution

It is difficult to obtain an analytical expression for the exact distribution of the circular Rao's spacing-frequencies test statistic given in (6.16). One can, however, obtain a formula for its characteristic function, along the lines of Bartlett (1938); see also Mirakhmedov et al. (2014). More generally, we consider the symmetric test statistic $T_{m,n}$ given in (6.5).

Consider the negative binomial random variables given in (6.12) with $l = 1$ and $\varphi : (\mathbb{R}^m, \mathsf{B}(\mathbb{R}^m)) \to (\mathbb{R}, \mathsf{B}(\mathbb{R}))$. Let $v \in \mathbb{R}$ and $k \in \mathbb{N}$, then

$$\mathsf{E}\left[\varphi(W_1, \ldots, W_m)\mathrm{e}^{\mathrm{i}vZ_m}\right] = \sum_{k=0}^{\infty}\mathrm{e}^{\mathrm{i}vk}\mathsf{P}[Z_m = k]\mathsf{E}[\varphi(W_1, \ldots, W_m) \mid Z_m = k]$$

(where $Z_m = \sum_{j=1}^{m} W_j$). The right side of the aforementioned equation is a Fourier series and from Fourier inversion we obtain

$$\mathsf{E}[\varphi(W_1,\ldots,W_m) \mid Z_m = k] = \frac{1}{2\pi \mathsf{P}[Z_m = k]} \int_{-\pi}^{\pi} \mathsf{E}[\varphi(W_1,\ldots,W_m)\mathrm{e}^{\mathrm{i}v(Z_m-k)}]\mathrm{d}v. \tag{6.18}$$

The conditional representation (6.13) directly yields

$$\mathsf{E}[\varphi(S_1,\ldots,S_m)] = \mathsf{E}[\varphi(W_1,\ldots,W_m) \mid Z_m = n],$$

which together with (6.18) at $k = n$ yields

$$\mathsf{E}[\varphi(S_1,\ldots,S_m)] = \frac{1}{2\pi \mathsf{P}[Z_m = n]} \int_{-\pi}^{\pi} \mathsf{E}[\varphi(W_1,\ldots,W_m)\mathrm{e}^{\mathrm{i}v(Z_m-n)}]\mathrm{d}v.$$

Define $\nu = \mathsf{E}[W_1] = (1-p)/p = \rho^{-1}$ and $\tau^2 = \mathrm{var}(W_1) = (1-p)/p^2 = (1+\rho)/\rho$. Given the function h of the symmetric test statistic given in (6.5) and $v_1, v_2 \in \mathbb{R}$, we define

$$\psi(v_1,v_2) = \mathsf{E}\left[\exp\left\{\mathrm{i}\frac{v_1}{\zeta_1}\left(h(W_1) - \mathsf{E}[h(W_1)] - \frac{\mathrm{cov}(h(W_1),W_1)}{\tau}(W_1-\nu)\right) + \mathrm{i}\frac{v_2}{\tau}(W_1-\nu)\right\}\right]$$

and

$$\hat{\psi}_m(v_1,x) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\tau\sqrt{m}}^{\pi\tau\sqrt{m}} \mathrm{e}^{-\mathrm{i}v_2 x}\psi^m(v_1,v_2)\mathrm{d}v_2,$$

for $x \in \mathbb{R}$, where ζ_1 is defined by (6.15). This last result and the inversion formula for the probability $\mathsf{P}[Z_m = n]$ provide a Bartlett-type formula for the characteristic function of

$$U_{m,n} = \frac{1}{\sigma_1}\sum_{j=1}^{m}\left\{h(S_j) - \mathsf{E}[h(W_1)] - \frac{\mathrm{cov}(h(W_1),W_1)}{\tau}(S_j-\nu)\right\},$$

which is given by

$$\mathsf{E}\left[\mathrm{e}^{\mathrm{i}vU_{m,n}}\right] = \frac{\hat{\psi}_n(v,x)}{\hat{\psi}_m(0,x)}. \tag{6.19}$$

Getting an analytical form for this characteristic function and inverting it to the exact probability is a difficult task, although asymptotic distribution and Edgeworth expansion can be obtained along the lines of Mirakhmedov et al. (2014).

However, given that one can compute the list of all possible realizations of $(S_1,\ldots,S_m)$, for any given $n \geq 1$, one can actually compute the value of the statistic for each of these $\binom{n+m-1}{n}$ equiprobable configurations and in this way determine the exact probability distribution of Rao's spacing-frequencies statistic $T_{n,m}$ given in (6.16).

6.3.3 Saddlepoint approximation

An alternative to finding all possible realizations of the spacing-frequencies is to approximate the exact distribution of Rao's spacing-frequencies statistic by the saddlepoint approximation. The saddlepoint approximation is a large deviations technique, which provides approximations to the exact distributions with bounded relative error. It is thus a very accurate method for computing small tail probabilities. It was introduced in statistics by Daniels (1954). In this section, we provide the cumulant generating function required for computing the saddlepoint approximation of Gatto and Jammalamadaka (1999) to the distribution of Rao's spacing-frequencies test statistics, under H_0.

For this purpose, we reexpress Rao's spacing-frequencies statistic (6.16) in the general M-statistic form $\sum_{j=1}^{m} \psi_1(S_j, T_{m,n}) = 0$, where

$$\psi_1(x, t_1) = \frac{1}{2}\left|\frac{n}{m} - x\right| - \frac{t_1}{m} = \begin{cases} \frac{1}{2}(\frac{n}{m} - x) - \frac{t_1}{m}, & \text{if } x \leq \frac{n}{m}, \\ \frac{1}{2}(x - \frac{n}{m}) - \frac{t_1}{m}, & \text{if } x > \frac{n}{m}. \end{cases}$$

We also define $\psi_2(x, t_2) = x - t_2/m$. Next, we compute the following joint cumulant generating function of these scores,

$$K(v_1, v_2; t_1, t_2) = \log\left\{\mathsf{E}[\exp\{v_1\psi_1(W_1, t_1) + v_2\psi_2(W_1, t_2)\}]\right\},$$

where W_1 has the distribution (6.12) with $l = 1$, which is a geometric distribution. After algebraic simplifications, we find

$$K(v_1, v_2; t_1, t_2) =$$

$$\log p + \log\left(\exp\left\{\frac{1}{m}\left[v_1\left(\frac{n}{2} - t_1\right) - v_2 t_2\right]\right\}\frac{1 - \left\{(1-p)e^{v_2 - \frac{v_1}{2}}\right\}^{\lfloor\frac{n}{m}\rfloor + 1}}{1 - (1-p)e^{v_2 - \frac{v_1}{2}}}\right.$$

$$\left. + \exp\left\{-\frac{1}{m}\left[v_1\left(\frac{n}{2} + t_1\right) + v_2 t_2\right]\right\}\frac{\left\{(1-p)e^{\frac{v_1}{2} + v_2}\right\}^{\lfloor\frac{n}{m}\rfloor + 1}}{1 - (1-p)e^{\frac{v_1}{2} + v_2}}\right),$$

$\forall v_1, v_2 \in \mathbb{R}$ such that $v_1/2 + v_2 < -\log(1-p)$. The saddlepoint approximation to $\mathsf{P}[T_{m,n} \geq t_1]$ can now be obtained by a direct application of Step 1 and Step 2 provided at p. 534 of Gatto and Jammalamadaka (1999) to the function $K_n = nK$, where K is the cumulant generating function given by the aforementioned formula. We also set $t_2 = n$ and $p = \rho_\nu/(\rho_\nu + 1)$, where $\rho_\nu = m/n$, see (6.6). One may refer to Gatto (2000) for a continuity correction for the case where the statistic is discrete, as it is the case here, and also for an algorithm for computing the quantiles, that is, the critical values of the test.

Although this approximation represents only the leading term of an asymptotic series, its accuracy is very good, even for small values of m and n and for very small tail probabilities. For the "exponential score" spacing-frequencies statistic, Table 1 in Gatto (2000) shows that with sample sizes as small as $m = 4$ and $n = 12$, this approximation is good: it yields 12% as relative error when applied to the upper tail probability 1%.

6.4 Monte Carlo power comparisons

This section presents a comparison of the power of Wheeler–Watson, Dixon's and Rao's spacing-frequencies tests, under a specific deviation from the null hypothesis, which appear unfavorable for Wheeler–Watson test. Numerical evaluations are done by Monte Carlo simulation, because it does not seem possible to extend the saddlepoint approximation of Section 6.3.3 to distributions under the alternative hypothesis. The reason is that the conditional representation (6.13) is valid only under the null hypothesis.

As is done on the real line, it is possible through a probability integral transform to make the distribution of say X uniform. Thus, let us consider the null hypothesis (6.1) wherein P_X is the circular uniform distribution with density $f_X(\theta) = 1/(2\pi)$, $\forall \theta \in \mathbb{R}$, and alternatives where and P_Y is a generalized von Mises distribution (GvM) of order two, with density given by

$$f_Y(\theta \mid \mu_1, \mu_2, \kappa_1, \kappa_2) = \frac{1}{2\pi G_0(\delta, \kappa_1, \kappa_2)} \exp\{\kappa_1 \cos(\theta - \mu_1) + \kappa_2 \cos 2(\theta - \mu_2)\}, \tag{6.20}$$

$\forall \theta \in \mathbb{R}$, where $\mu_1 \in [0, 2\pi)$, $\mu_2 \in [0, \pi)$, $\kappa_1, \kappa_2 \geq 0$, $\delta = (\mu_1 - \mu_2) \mathrm{mod} \pi$ and where the normalizing constant is given by

$$G_0(\delta, \kappa_1, \kappa_2) = \frac{1}{2\pi} \int_0^{2\pi} \exp\{\kappa_1 \cos \theta + \kappa_2 \cos 2(\theta + \delta)\} d\theta.$$

A circular random variable with density (6.20) is denoted $\mathrm{GvM}(\mu_1, \mu_2, \kappa_1, \kappa_2)$. We refer to Gatto and Jammalamadaka (2007) for various interesting theoretical properties and characterizations regarding this class of distributions. We note that the well-known von Mises distribution is obtained by setting $\kappa_2 = 0$ in (6.20) and that the uniform distribution (with density f_X) is obtained by setting $\kappa_1 = \kappa_2 = 0$ in (6.20).

We consider alternative hypotheses where P_Y is the GvM distribution with $\mu_1 = \mu_2 = 0$, $\kappa_1 = 0.1$ and $\kappa_2 \in \{0.5, 1, \ldots, 7\}$. The graphs of some of these densities, over the interval $[-\pi, \pi)$, are given in Figure 6.1. We can see that each density is symmetric around zero and possesses two clear and quite similar modes. Figure 6.1 shows also that these GvM

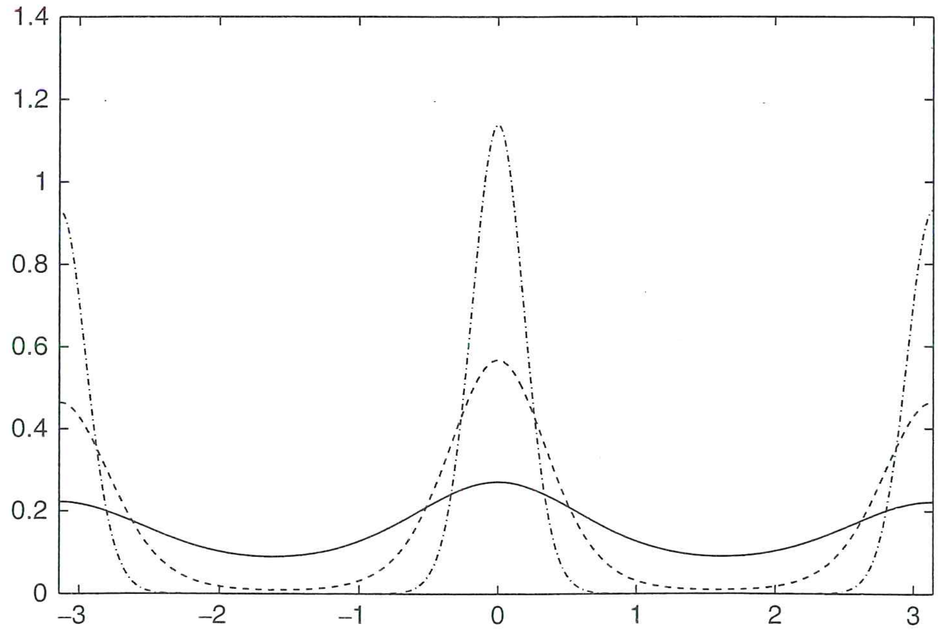

Figure 6.1 GvM densities over $[-\pi, \pi)$ with $\mu_1 = \mu_2 = 0$, $\kappa_1 = 0.1$ and $\kappa_2 = 0.5$ (solid line), $\kappa_2 = 2$ (dashed line), $\kappa_2 = 7$ (dashed-dotted line).

densities deviate increasingly from uniformity as the value of κ_2 increases. We compare the small sample power of the following tests: Wheeler–Watson test (see Example 7), Dixon's test (see Example 5) and Rao's spacing-frequencies test (see Section 6.3.1). All tests have (approximate) size 5% and the selected sample sizes are $m = 15$ and $n = 25$. Let us rewrite Wheeler–Watson test statistic $||V_X||$ given in (6.9) as $T^W_{m,n}$ and let us denote its α^{th} upper tail quantile as t^W_α. Let us also rewrite Dixon's spacing-frequencies test statistic $T_{m,n}$ given in (6.8) as $T^D_{m,n}$ and let us denote its α^{th} upper tail quantile as t^D_α. Let us also rewrite Rao's spacing-frequencies test statistic $T_{m,n}$ given in (6.16) as $T^R_{m,n}$ and let us denote its α^{th} upper tail quantile as t^R_α. Large values of $T^W_{m,n}$, $T^D_{m,n}$, and $T^R_{m,n}$ provide evidence against H_0. Based on $0.5 \cdot 10^6$ Monte Carlo simulations, we obtain $t^W_{0.05} = 5.3305$, $t^D_{0.05} = 149$, and $t^R_{0.05} = 14.6667$. In this setting, the powers of Wheeler–Watson and Rao's tests have been computed for various values of κ_2, each time based on 10^5 Monte Carlo generations.

The results are displayed in Table 6.1. We see that Wheeler–Watson test appears substantially less powerful for distinguishing the uniform distribution from the selected bimodal GvM distributions. This confirms the claim given at the end of Example 7, that the Wheeler–Watson test may not be appropriate when dealing with bimodal distributions displaying two similar well-separated modes with one at the antimode. Dixon's and Rao's spacing-frequencies test behave substantially better in this case. For other configurations with less accentuated bimodality, the power of Wheeler–Watson test is closer to the one of its competitors. Nevertheless, this important result and conclusion are in the same spirit as the well-known result that the Rayleigh test in one-sample case loses to tests such as the one-sample Rao's spacings test and is indeed inappropriate, when the data is not unimodal. We see also that Dixon's test shows slightly better power than Rao's test when κ_2 is small, that is, close to the null hypothesis and is known to be asymptotically locally most

Table 6.1 Power comparison between Wheeler–Watson, Dixon's and Rao's tests.

κ_2	$\mathsf{P}_{\kappa_2}[T^W_{m,n} > t^W_{0.05}]$	$\mathsf{P}_{\kappa_2}[T^D_{m,n} > t^D_{0.05}]$	$\mathsf{P}_{\kappa_2}[T^R_{m,n} > t^R_{0.05}]$
0.5	0.060	0.090	0.056
1.0	0.074	0.189	0.142
1.5	0.090	0.326	0.291
2.0	0.104	0.462	0.456
2.5	0.117	0.563	0.588
3.0	0.127	0.641	0.684
3.5	0.134	0.670	0.754
4.0	0.141	0.743	0.804
4.5	0.146	0.776	0.841
5.0	0.151	0.803	0.866
5.5	0.153	0.824	0.887
6.0	0.157	0.843	0.903
6.5	0.161	0.859	0.918
7.0	0.161	0.870	0.928

P_X: uniform distribution. P_Y: GvM distribution with $\mu_1 = \mu_2 = 0$, $\kappa_1 = 0.1$ and $\kappa_2 = 0.5, 1, \ldots, 7$. Each probability is obtained from 10^5 simulations. Size of tests: 5%. $m = 15$, $n = 25$.

powerful test among the symmetric tests in (6.5). However, this small advantage turns in favor of Rao's test as κ_2 increases.

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